8.2 Cyclic coordinates and conservation theorems.

- cyclic coordinate q_j : does not appear in the Lagrangian
  $\Rightarrow$ constant conjugate momentum

Recall:

- GC and GM (generalized momenta) or CM (conjugate momenta)
  q_i p_i

$$\boxed{p_i = \frac{\partial L(q_j, \dot{q}_j, t)}{\partial \dot{q}_i}} \qquad (8.2)$$

- Substituting in LEs:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \qquad (8.14)$$

- Hamiltonian $\boxed{H = \dot{q}_i p_i - L(q_i, \dot{q}_i, t)}$ (8.15)

which had a differential: $H(\dot{q}_i, p)$

$$dH = \dot{q}_i \, dp_i - \dot{p}_i \, dq_i - \frac{\partial L}{\partial t} dt \qquad (8.16)$$

- (8.14) $\leftrightarrow$ (8.16):

$$\dot{p}_i = \frac{\partial L}{\partial q_i} = -\frac{\partial H}{\partial q_i}$$

$\Rightarrow$ a cyclic coordinate also absent from H.

- Note H differs from $-L$ only by $p_i \dot{q}_i$, which does not involve q_i explicitly.

- If a GC does not appear in H
  $\Rightarrow$ conjugate momentum conserved.

- Momentum conservation theorems can be transferred to the Hamiltonian formalism by substitution of H for L

- Connection btw the invariance and symmetry properties of the physical system and the constants of the motion can be derived in terms of H.
- Discussion of h energy function (section 2.7)
- If L (and thus H → 8.15) is not an explicit function of time,

  then H : constant of motion.

  → can be seen from

$$\left.\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i} \\ -\dot{p}_i &= \frac{\partial H}{\partial q_i} \end{aligned}\right\} \text{ Canonical equations of Hamilton} \qquad (8.18)$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t}$$

by writing the total time derivative of H:

$$dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt \qquad H = H(p_i, q_i, t)$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial q_i} \dot{q}_i + \frac{\partial H}{\partial p_i} \dot{p}_i + \frac{\partial H}{\partial t}$$

$$= \cancel{-\dot{p}_i \dot{q}_i} + \cancel{\dot{q}_i \dot{p}_i} - \frac{\partial L}{\partial t}$$

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t} \qquad (8.41)$$

⇒ If t doesn't appear explicitly in L, it will also not be present in H. → H constant in time

- Recall: if the eqn's of transformation that define the GCs:

$$\vec{r}_m = \vec{r}_m(q_1, \dots, q_n, t) \qquad (1.38)$$

do not depend explicitly on time and
if the potential is velocity independent,
then $H = T + V$ is the total energy.

- Note that identification of H
  as a constant of the motion and
  as the total energy are two separate matters.

Hamiltonian depends both in magnitude and in functional form upon the initial choice of GCs.

Lagrangian: specific prescription $L = T - V$.

- change of GCs may change the functional appearance of L but not its magnitude
- but different GCs $\longrightarrow$ different quantity for H.
- possible that for one set of GCs H is conserved, for another it varies with time.

Example: artificial 1D system.

- point mass attached to a spring, of force constant k, the other end of which is fixed on a massless cart that is being moved uniformly by an external device with speed v_0.

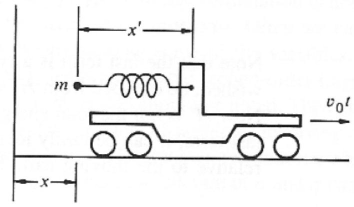


FIGURE 8.1 A harmonic oscillator fixed to a uniformly moving cart.

- GC : position x of the mass particle in the stationary system, then

$$L = L(x, \dot{x}, t) = T - V$$
$$= \frac{m\dot{x}^2}{2} - \frac{k}{2}(x - v_0 t)^2 \qquad (8.42)$$

- eom : $m\ddot{x} = -k(x - v_0 t)$
- how to solve? :

$$x' = x - v_0 t$$ displacement of the particle relative to the cart.
$$\ddot{x}' = \ddot{x}$$

$$\Rightarrow \quad m\ddot{x} = -kx'$$

- To an observer on the cart, the particle exhibits simple harmonic motion.
(expected on the principle of equivalence in Galilean relativity)

- Hamiltonian approach :
  - x : Cartesian coordinate of the particle
  - Potential does not involve GVs

  $H = T + V$: total energy
  $$= \frac{p^2}{2m} + \frac{k}{2}(x - v_0 t)^2$$
  - H explicit function of time $\rightarrow$ not conserved

  Physics : Energy must flow in and out of the external physical device to keep the cart moving uniformly, against the reaction of the oscillating particle.

- Lagrangian

Suppose: we formulated in terms of the relative coordinate x'.

$$\Rightarrow L = L(x, \dot{x}') = \frac{m\dot{x}'^2}{2} + m\dot{x}'v_0 + \frac{mv_0^2}{2} - \frac{kx'^2}{2}$$

- Corresponding Hamiltonian:

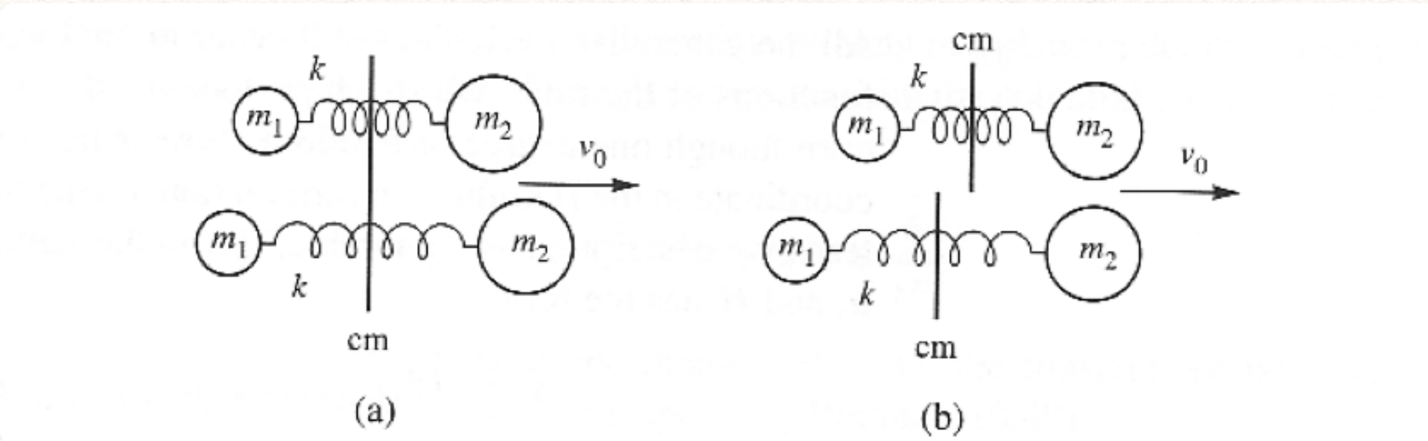
$$H' = H'(x', p') = \underbrace{\frac{(p' - mv_0)^2}{2m} + \frac{kx'^2}{2}}_{\text{Total energy of the system}} - \underbrace{\frac{mv_0^2}{2}}$$

constant, involving neither x' nor p', so drop it from H' without affecting eom

- H' not the total energy but conserved!
- H and H' different in magnitude, time dependence and functional behavior, but both lead to the same motion of the particle.

- A dumbbell of two masses connected by a spring of constant k,
- Consider the com in constant motion at speed v_0 along the direction determined by the spring and allow oscillations only along this direction.
- Make the dumbbell to vibrate while its com has an initial v_0.
- It will continue with this velocity with uniform translational motion.
- The translation motion – no effect on the oscillation

- Com motion and motion relative to com separate
- Start of the motion, then E conserved and $H = E_{tot}$
- Different situation :
  - If m_2 moving at constant speed v_0, since a periodic force is applied.
  - Then m_1 and com oscillate relative to m_2.
  - Since a changing external force must be applied to the system to keep m_2 at v_0, H no longer conserved ; $H \neq E_{tot}$.



8.3 Routh's procedure

- Hamiltonian procedure especially adapted to problems involving cyclic coordinates.

$$L = L(q, \dot{q}, t) = L(q_1, \dots, q_{n-1}, \dot{q}_1, \dots, \dot{q}_{n-1}, t)$$

- still need to solve a problem of n degrees of freedom, even though one dof corresponds to a cyclic coordinate.
- Cyclic coordinate in Hamiltonian formulation is "ignorable", because p_n is some constant α:

$$H = H(q_1, \dots, q_{n-1}; p_1, \dots, p_{n-1}, \alpha; t)$$

- H now describes a problem involving $n-1$ coordinates
  → may be solved ignoring the cyclic coordinate, except as it is manifested in the constant of integration α.
- Behavior of cyclic coordinate from:

$$\dot{q}_n = \frac{\partial H}{\partial \alpha}$$

Routh method:

- Advantages of H-formulation in handling cyclic c. +
  L- —"— noncyclic c.
  - mathematical transformation from $q, \dot{q}$ basis to the q, p basis only for cyclic c., obtaining their eom in the Hamiltonian form,
  - remaining coordinates governed by LEs.

- Cyclic c.: $q_{s+1}, \dots, q_n$

New function R, Routhian:

$$R(q_1, \dots, q_n; \dot{q}_1, \dots, \dot{q}_s; p_{s+1}, \dots, p_n; t) = \sum_{i=s+1}^{n} p_i \dot{q}_i - L$$

$$= H_{cyc}(p_{s+1}, \dots, p_n) - L_{noncycl}(q_1, \dots, q_s; \dot{q}_1, \dots, \dot{q}_s)$$

- For the s-ignorable coordinates

$$\frac{d}{dt}\left(\frac{\partial R}{\partial \dot{q}_i}\right) - \frac{\partial R}{\partial q_i} = 0 \qquad i = 1, \dots, s \qquad (8.50)$$

For $n-s$ ignorable coordinates, Hamilton's eqn's apply:

$$\frac{\partial R}{\partial q_i} = -\dot{p}_i = 0 \quad \text{and} \quad \frac{\partial R}{\partial p_i} = \dot{q}_i \qquad i = s+1, \dots, n \qquad (8.51)$$

Example: Kepler problem (section 3.7)

A single particle moving in a plane under the influence of the inverse-square central force $f(r)$ derived from

$$V(r) = -\frac{k}{r^n}$$

- Lagrangian:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r^n}$$

Ignorable coordinate: θ

constant conjugate momentum p_θ

- Routhian:

$$R(r, \dot{r}, p_\theta) = \frac{p_\theta^2}{2mr^2} - \frac{1}{2}m\dot{r}^2 - \frac{k}{r^n}$$

- Routhian : equivalent 1D potential V' minus kinetic energy of radial motion.
- Apply LEs (8.50) to the noncyclic coordinate, the radial coord. r

$\Rightarrow$ eqn (3.11) :

$$\ddot{r} - \frac{p_\theta^2}{mr^3} + \frac{nk}{r^{n+1}} = 0$$

- Apply HE (8.51) to the cyclic variable θ

$$\dot{p}_\theta = 0 \quad \text{and} \quad \frac{p_\theta}{mr^2} = \dot{\theta}$$

same as (3.8) :

$$p_\theta = mr^2\dot{\theta} = l = \varsigma$$

- Routh's procedure
  - more automatic analysis
  - advantages in problems with many dof.