Chapter 2: Magnetohydrodynamics
Magnetohydrodynamic Turbulence
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Introduction

- Magnetohydrodynamics (MHD) describes the macroscopic behavior of a plasma
  - Macroscopic means acting on length scales larger than plasma lengths scales (i.e. Debye Length or Larmor radius)
- At the end of the chapter non-dimensional equations in terms of the Alfven time will be arrived at, which will be used throughout the rest of the book

MHD Equations-Momentum Terms

- To derive the momentum equation, the forces acting on a fluid element δV with mass $\rho \delta V$
- Lorentz Force: $q_j(\boldsymbol{E} + \boldsymbol{v}_j \times \boldsymbol{B}/c) \xrightarrow[q \approx 0]{\text{Quasi-Neutrality:}} \delta V \frac{1}{c} \boldsymbol{j} \times \boldsymbol{B},$
  - On macroscopic fields, charge neutrality is enforced
- Thermal Pressure Force: $-\oint p \, d\boldsymbol{S} = -\delta V \, \nabla p$
  - Assume approximate thermodynamic equilibrium so pressure tensor is isotropic and force is surface integral of fluid element
- Gravitational Force: $\delta V \, \rho \boldsymbol{g}$ $\quad \boldsymbol{g} = -\nabla \phi_g$

MHD Equations-Momentum Terms

- Viscous Force: $\oint \boldsymbol{\sigma}^{(\mu)} \cdot d\boldsymbol{S} = \delta V\, \nabla \cdot \boldsymbol{\sigma}^{(\mu)}$ $\quad \sigma_{ij}^{(\mu)} = \mu\big[(\partial_i v_j + \partial_j v_i) - \frac{2}{3}\delta_{ij}\, \nabla \cdot \boldsymbol{v}\big]$,
  - Another surface integral where **σ** is the viscous-stress tensor $\sigma_{ij}^{(\mu)} = \mu\big[(\partial_i v_j + \partial_j v_i) - \frac{2}{3}\delta_{ij}\, \nabla \cdot \boldsymbol{v}\big]$
- All of these forces acting on the same fluid element are added to give the momentum equation

$$\rho \frac{d\boldsymbol{v}}{dt} \equiv \rho(\partial_t + \boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B} + \rho \boldsymbol{g} + \mu(\nabla^2 \boldsymbol{v} + \tfrac{1}{3}\nabla\nabla \cdot \boldsymbol{v})$$

Pressure and Induction

- The Lorentz Force can also be written as

$$\frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} = -\frac{1}{8\pi}\nabla B^2 + \frac{1}{4\pi}\boldsymbol{B}\cdot\nabla\boldsymbol{B} = -\nabla\cdot\mathcal{T}^M,$$

where $\mathcal{T}^M = \{T_{ij}^M\}$ is the magnetic stress tensor,

$$T_{ij}^M = \frac{1}{8\pi}B^2\delta_{ij} - \frac{1}{4\pi}B_i B_j.$$

- The first term in the stress tensor is isotropic and together with thermal pressure gives the total pressure

$$P = p + B^2/(8\pi)$$

- The ratio between these terms is the important plasma β parameter

$$\beta = 8\pi p/B^2$$

Magnetic Induction Equation

- The time derivative of the magnetic field is given by Faraday's Law

$$\partial_t \boldsymbol{B} = -c\,\nabla \times \boldsymbol{E}$$

- Where the Electric field can be obtained from Ohm's Law $\boldsymbol{E} + \frac{1}{c}\boldsymbol{v} \times \boldsymbol{B} = \frac{1}{\sigma}\boldsymbol{j}$
- And the magnetic field can be substituted for current from Ampere's Law **j**=∇ x **B**, this leaves a ∇ x (**v** x **B**) and a ∇ x (∇x **B**) which yields

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \eta\,\nabla^2 \boldsymbol{B}.$$

Conservation of Mass

- Within a fluid element, the change of mass will be given by the flow in and out of the surface of the element

$$\frac{d}{dt}\int_{\mathcal{V}} \rho \, dV = -\oint \rho \boldsymbol{v} \cdot d\mathbf{S}.$$

- Using Gauss' theorem on the surface integral:

$$\partial_t \rho + \nabla \cdot \rho \boldsymbol{v} = \frac{d\rho}{dt} + \rho \nabla \cdot \boldsymbol{v} = 0.$$

- We will deal largely with incompressible flow, so the mass density of an element is constant

Dynamic Equation for Pressue

- Assuming the plasma is close to thermodynamic equilibrium pressure is coupled to ρ and T by the equation of state
- We will assume the ideal gas law is valid for dilute plasmas $p = 2(\rho/m_i)k_B T = 2nk_B T$
- Also, conduction will be neglected so the change of state can be assumed to be adiabatic

$$d(p\rho^{-\gamma})/dt = 0 \longrightarrow \partial_t p + \boldsymbol{v}\cdot\nabla p + \gamma p \nabla\cdot\boldsymbol{v} = 0.$$

- In the limit of incompressible flow, pressure can be curled out of dynamic equations (more later)

Pressure (Cont.)

- γ is the adiabatic exponent, given by the ratio of specific heats
- If heat conduction is not negligible the temperature is given by $\partial_t T + \boldsymbol{v} \cdot \nabla T + (\gamma - 1) T \nabla \cdot \boldsymbol{v} = (\gamma - 1) \kappa \nabla^2 T$

  where κ is the diffusivity. This formula is a very simplified form of the processes going on in plasma embedded in a magnetic field
- The heat equation, if both conduction and dissipation are considered, is given by

$$\rho(\partial_t + \boldsymbol{v} \cdot \nabla) u + p \nabla \cdot \boldsymbol{v} = -\nabla \cdot \boldsymbol{q} + \boldsymbol{\sigma}^{(\mu)} : \nabla \boldsymbol{v} - \frac{1}{\sigma} \boldsymbol{j}^2$$

with the heat flux $\boldsymbol{q} = -\kappa \rho \nabla T$ (the colon denotes the dyadic product).

Rotating Reference Frame

- To conserve angular momentum, astrophysical objects rotate
- To describe these systems, a co-rotating reference frame is useful

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r} \qquad \boldsymbol{v} \equiv \frac{d\boldsymbol{r}}{dt} = \frac{d\boldsymbol{r}}{dt'} + \boldsymbol{\Omega} \times \boldsymbol{r}$$

$$\frac{d\boldsymbol{v}}{dt} = \frac{d\boldsymbol{v}}{dt'} + \boldsymbol{\Omega} \times \boldsymbol{v}$$

$$= \frac{d\boldsymbol{u}}{dt'} + 2\boldsymbol{\Omega} \times \boldsymbol{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}).$$

- Substituting dv/dt from the momentum equation

$$\rho(\partial_{t'} + \boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B} + \rho \boldsymbol{g}$$

$$+ \rho\Big[2\boldsymbol{u} \times \boldsymbol{\Omega} + \boldsymbol{\Omega} \times (\boldsymbol{r} \times \boldsymbol{\Omega})\Big] + \mu \nabla^2 \boldsymbol{u}.$$

- The other dynamic equations (**B** and ρ) remain the same in the rotating reference frame

Incompressibility

- In most work with turbulence, the fluid is considered incompressible
- Conditions for incompressibility
  - Fluid motion must be slow compared to the fastest propagating compressible wave in the direction of motion
  - Small temporal derivative
- $\nabla \bullet v = 0$, so taking the curl of the motion equation leads to the vorticity form where $\boldsymbol{\omega} = \nabla \times \boldsymbol{v},$

$$\partial_t \boldsymbol{\omega} + \boldsymbol{v} \cdot \nabla \boldsymbol{\omega} - \boldsymbol{\omega} \cdot \nabla \boldsymbol{v} = \frac{1}{c\rho_0}(\boldsymbol{B} \cdot \nabla \boldsymbol{j} - \boldsymbol{j} \cdot \nabla \boldsymbol{B}) + \nu \nabla^2 \boldsymbol{\omega}.$$

Incompressibility (Cont.)

- If the vorticity is calculated, it can be converted to velocity via Poisson's equation

$$\nabla^2 \boldsymbol{v} = -\nabla \times \boldsymbol{\omega}.$$

- The pressure is now curled out of the equation, and is now a calculated quantity rather than an independent variable

$$\nabla^2 P = -\nabla \cdot [\rho_0 \boldsymbol{v} \cdot \nabla \boldsymbol{v} - \boldsymbol{B} \cdot \nabla \boldsymbol{B}/(4\pi)].$$

Compressibility Conditions

- For high-β plasma, the momentum equation becomes $$\nabla \cdot \boldsymbol{v} \simeq \frac{\rho}{\gamma p} \boldsymbol{v} \cdot \nabla \left(\frac{1}{2} v^2 \right) \sim \mathrm{M}_s^2 \frac{v}{L},$$

$$\mathrm{M}_s = v / c_s$$

- M_s is the accoustic Mach Number, c_s is the sound speed given by $, c_s = \sqrt{\gamma p / \rho}$ and L is a gradient scale.
- The fluid motion is incompressible if $M_s << 1$

Compressibility Conditions

- For low-β plasma the Magnetic pressue dominates motion and

$$\nabla \cdot \boldsymbol{v} \simeq \frac{4\pi\rho}{B^2}\boldsymbol{v} \cdot \nabla\left(\tfrac{1}{2}v^2\right) \sim \mathrm{M}_{\mathrm{A}}^2 \frac{v}{L}$$

$$\mathrm{M}_{\mathrm{A}} = v/v_{\mathrm{A}}$$

- M_A is the Alfven Mach Number, and v_A is the Alfven speed speed given by $v_{\mathrm{A}} = B/\sqrt{4\pi\rho_0}$
- Again if M_A << 1 the and the motion is perpendicular to a strong magnetic field, the fluid is incompressible
- Essentially, regardless of whether the Thermal or Magnetic pressure is dominant, if the flow velocity is much less than the characteristic wave speed the fluid is compressible

Boussinesq Approximation

- In a sufficient gravity, pressure and density are in hydrostatic equilibrium $\nabla p_0 = \mathbf{g}\rho_0$ and the magnetic field is negligible.
- The gradient in the hydrostatic equilibrium equation provides a typical length scale, focusing on a layer of plasma much smaller than this, the system can be considered approximately homogeneous
- This quasi-homogeneity means the deviations of the dynamic variables $\tilde{\rho}, \tilde{p}, \tilde{\boldsymbol{B}}$ do not significantly differ from their equilibrium values and is the bases of the Boussinesq Approximation

Boussinesq Approximation

- The differences in the dynamic variables are still finite however, so the motion equation still has non-linear terms

$$\partial_t \boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} = -\frac{1}{\rho_0}\nabla\widetilde{p} + \frac{1}{c\rho_0}\widetilde{\boldsymbol{j}}\times(\boldsymbol{B}_0 + \widetilde{\boldsymbol{B}}) + \frac{1}{\rho_0}\boldsymbol{g}\widetilde{\rho} + 2\boldsymbol{v}\times\boldsymbol{\Omega} + \nu\nabla^2\boldsymbol{v}$$

$$\partial_t \boldsymbol{\omega} + \boldsymbol{v}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\boldsymbol{v} = \frac{1}{c\rho_0}[\boldsymbol{B}_0\cdot\nabla\widetilde{\boldsymbol{j}} - \widetilde{\boldsymbol{j}}\cdot\nabla(\boldsymbol{B}_0 + \widetilde{\boldsymbol{B}})] + \frac{1}{\rho_0}\nabla\widetilde{\rho}\times\boldsymbol{g} + 2\boldsymbol{\Omega}\cdot\nabla\boldsymbol{v} + \nu\nabla^2\boldsymbol{\omega}$$

- Considering perturbations acting on hydrostatic equilibrium and remember $L/L_G << 1$, the pressure change can be shown to be negligible

$$\frac{\widetilde{p}}{p_0} \sim \frac{L}{L_g}\frac{\widetilde{\rho}}{\rho_0} \ll \frac{\widetilde{\rho}}{\rho_0}$$

Boussinesq Approximation

- Using the ideal-gas law, we can now relate the pressure fluctuation to the temperature fluctuation

$$\frac{\tilde{p}}{p_0} = \frac{\tilde{\rho}}{\rho_0} + \frac{\tilde{T}}{T_0}$$

- For incompressible motions, the temperature fluctuation is given by $\partial_t \tilde{T} + \boldsymbol{v} \cdot \nabla(\tilde{T} + T_0) = \kappa \, \nabla^2 \tilde{T}$

MHD in 2D

$$\boldsymbol{v} = \boldsymbol{e}_z \times \nabla\phi, \quad \boldsymbol{B} = \nabla \times A_z \boldsymbol{e}_z = \boldsymbol{e}_z \times \nabla\psi.$$

$$\omega = \omega_z = \nabla^2\phi, \quad j = j_z = \frac{c}{4\pi}\nabla^2\psi$$

$$\partial_t\omega + \boldsymbol{v}\cdot\nabla\omega = [1/(c\rho_0)]\boldsymbol{B}\cdot\nabla j + \nu\nabla^2\omega$$

$$\partial_t\psi + \boldsymbol{v}\cdot\nabla\psi = \eta\nabla^2\psi$$

Conservation Laws-Fluid Invariants

- The momentum equation in conservation form

$$\partial_t \rho\, \boldsymbol{v} = \nabla \cdot \mathcal{T} + \rho \boldsymbol{g},$$

where $\mathcal{T} = \{T_{ij}\}$ is the total stress tensor, which can be written in the form

$$T_{ij} = -\left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \left(\rho v_i v_j - \frac{B_i B_j}{4\pi}\right) + \mu(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}\delta_{ij}\, \nabla \cdot \boldsymbol{v})$$
$$= -P\delta_{ij} + R_{ij} + \sigma_{ij}^{(\mu)}.$$

$$\frac{d}{dt}\int_V \rho \boldsymbol{v}\, dV = \oint_S \mathcal{T} \cdot d\boldsymbol{S} + \int_V \rho \boldsymbol{g}\, dV$$

- Total Energy Density

$$\partial_t\left(\rho(\tfrac{1}{2}v^2 + u + \phi_g) + \frac{1}{8\pi}B^2\right) + \nabla \cdot \boldsymbol{F}^E = 0,$$

with the energy flux

$$\boldsymbol{F}^E = (\tfrac{1}{2}v^2 + h + \phi_g)\rho \boldsymbol{v} - \boldsymbol{\sigma}^{(\mu)} \cdot \boldsymbol{v} + \boldsymbol{q} + \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B}$$

where h is the enthalpy,

$$h = u + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

The Energy Equation

- Integrating the energy equation:

$$\frac{dE}{dt} = -\oint_S d\boldsymbol{S} \cdot \boldsymbol{F}^E - D^E$$

$$E = \int dV \left(\frac{1}{2}\rho v^2 + \rho\phi_g + \frac{1}{8\pi} B^2 \right)$$

$\boldsymbol{F}^E$ contains no dissipative terms,

$$\boldsymbol{F}^E = (\tfrac{1}{2}\rho v^2 + p + \rho\phi_g)\boldsymbol{v} + \frac{1}{4\pi}\boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B})$$

and D^E comprises the energy dissipation,

$$D^E = \int dV \left(\frac{1}{\sigma}\boldsymbol{j}^2 + \mu\boldsymbol{\omega}^2 \right).$$

Cross-Helicity

- Another invariant is the cross-helicity

$$H^C = \int \boldsymbol{v} \cdot \boldsymbol{B} \, dV$$

$$\frac{dH^C}{dt} = -\oint_S \boldsymbol{F}^C \cdot d\boldsymbol{S} - D^C$$

- with cross-helicity flux

$$\boldsymbol{F}^C = \boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{B}) + \left(\phi_g + \frac{\gamma}{\gamma - 1} \frac{p}{\rho}\right) \boldsymbol{B}$$

- and cross-helicity dissipation

$$D^C = (\nu + \eta) \int_V dV \sum_{i,j} \partial_i B_j \, \partial_i v_j$$

Magnetic Invariants-Flux

- Magnetic Flux defined by $\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$
- Applying Stokes' Theorem to the induction eq.:

$$\int_S \partial_t \boldsymbol{B} \cdot d\boldsymbol{S} = \oint_l (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} - \frac{c}{\sigma} \oint_l \boldsymbol{j} \cdot d\boldsymbol{l}.$$

$$\oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}\, dt = -\oint \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{l})\, dt = \int_{dS} \boldsymbol{B} \cdot d\boldsymbol{S}$$

- So the time derivative of the flux is

$$\frac{d\Phi}{dt} = \int_S \partial_t \boldsymbol{B} \cdot d\boldsymbol{S} + \oint_l \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{l}) = -\frac{c}{\sigma} \oint_l \boldsymbol{j} \cdot d\boldsymbol{l}.$$

- The conductivity is assumed to be infinite, so the magnetic flux is conserved

Magnetic Invariants-Helicity

- Since the magnetic field is frozen into the plasma, the field is complex. The complexity is given by the helicity $H^M = \int_V \mathbf{A} \cdot \mathbf{B} \, dV,$
- To make sure, the helicity is gauge invariant

$$\mathbf{A}' = \mathbf{A} + \nabla \chi$$

$$H^{M'} - H^M = \int_V \mathbf{B} \cdot \nabla \chi \, dV = \oint_S \chi \mathbf{B} \cdot d\mathbf{S}.$$

- In some cases this helicity is gauge invariant, but in actual cases such as open field lines extending to the solar wind and and bound to the photosphere, this helicity is not gauge invariant. An alternative helicity was proposed

$$H^M_{\mathrm{alt}} = \int_V dV \, (\mathbf{A} + \mathbf{A}_0) \cdot (\mathbf{B} - \mathbf{B}_0).$$

Helicity

- Inserting Farday's Law into the helicity equation:

$$\int \partial_t(\boldsymbol{A}\cdot\boldsymbol{B})\,dV = \int (\boldsymbol{B}\cdot\partial_t\boldsymbol{A} + \boldsymbol{A}\cdot\partial_t\boldsymbol{B})\,dV$$
$$= -2c\int \boldsymbol{E}\cdot\boldsymbol{B}\,dV + c\oint(\boldsymbol{A}\times\boldsymbol{E})\cdot d\boldsymbol{S}$$

- and then and using the boundary condition B_n=0 and Ohm's Law:

$$\oint(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{S}\,dt = \int_{dV}\boldsymbol{A}\cdot\boldsymbol{B}\,dV$$
$$\frac{dH^M}{dt} = \int \partial_t(\boldsymbol{A}\cdot\boldsymbol{B})\,dV + \oint(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{S} = -\frac{2c}{\sigma}\int dV\,\boldsymbol{j}\cdot\boldsymbol{B}.$$

- which again is 0 for infinite conductivity

2D Invariants

$$E = \int d^2x \left(\frac{1}{2}\rho v^2 + \frac{B^2}{8\pi} \right) = \int d^2x \left(\frac{1}{2}\rho(\nabla\phi)^2 + \frac{1}{8\pi}(\nabla\psi)^2 \right)$$

$$H^C = \int d^2x\, \boldsymbol{v}\cdot\boldsymbol{B} = -\int d^2x\, \omega\psi$$

$$A = \int d^2x\, \psi^2$$

Equilibrium Configurations

- While the book deals with turbulence, equilibrium is still important to consider
- In equilibrium, **v**=0 and the motion equation reduces to $\nabla p = \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B} + \rho \boldsymbol{g}$
- In the case of a strong magnetic field, gravity is negligible, so the equation of motion is just **j** x **B** with the pressure defined by magnetic pressure $P = p + B^2/(8\pi) = \text{constant}$ in a plane
- A specific magnetic equilibrium of interest in astrophysical plasma is a force free field when **j** x **B** = 0. If **j**=λ**B** it is a linear force free field

Equilibrium in Absence of Strong Magnetic Field

- In the absence of a strong magnetic field a pressure gradient $\nabla p = \rho \mathbf{g}$ and for polytropic pressure $p = p_0(\rho/\rho_0)^\gamma$ and assuming $\mathbf{g} = -g\mathbf{e}_z$

$$\rho(z) = \rho_0\left(1 - \frac{\gamma - 1}{\gamma}\frac{z}{L_g}\right)^{1/(\gamma-1)}$$

- When γ=1, this reduces to the barometric density profile

$$\rho(z) = \rho_0 e^{-z/L_g}$$

Linear Waves in a Homogeneous Magnetized System

- The basic elements of turbulence are formed by waves in the plasma, oscillating about an average state
- In a homogeneous system defined by p_0, ρ_0, embedded in magnetic field $\mathbf{B}_0$, for small changes $\tilde{p} \ll p_0, \tilde{\boldsymbol{b}} \ll \boldsymbol{B}_0$ the MHD equations are linear
- Performing a Fourier transform $\tilde{\boldsymbol{v}}(\boldsymbol{x}, t) = \boldsymbol{v}_1 \exp(i\boldsymbol{k} \cdot \boldsymbol{x} - i\varpi t)$ in space and time yields

$$-i\varpi\rho_0\boldsymbol{v}_1 = -i\boldsymbol{k}p_1 + \frac{1}{4\pi}(i\boldsymbol{k} \times \boldsymbol{b}_1) \times \boldsymbol{B}_0 - \mu k^2\boldsymbol{v}_1$$

$$-i\varpi\boldsymbol{B}_1 = i\boldsymbol{k} \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) - \eta k^2\boldsymbol{B}_1,$$

$$-i\varpi p_1 = -i\gamma p_0\boldsymbol{k} \cdot \boldsymbol{v}_1.$$

Waves in a Homogeneous Mangetized System (Cont.)

- Performing substitutions on v_1, B_1, and p_1, these three equations become one equation

$$\varpi^2 \rho_0 \boldsymbol{v}_1 = \left(\frac{\boldsymbol{B}_0 \times (\boldsymbol{k} \times \boldsymbol{B}_0)}{4\pi} + \gamma p_0 \boldsymbol{k} \right) \boldsymbol{k} \cdot \boldsymbol{v}_1 - \frac{1}{4\pi} \boldsymbol{k} \cdot \boldsymbol{B}_0 (\boldsymbol{k} \times \boldsymbol{v}_1) \times \boldsymbol{B}_0$$

- The **k·v** term represents longitudinal, compressible waves and the **k** x **v** term represents transverse waves
- By choosing the coordinate system $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z, \boldsymbol{k} = k_\perp \boldsymbol{e}_y + k_\parallel \boldsymbol{e}_z$ and writing the equation in matrix form

$$\begin{pmatrix} \varpi^2 - k_\parallel^2 v_A^2 & 0 & 0 \\ 0 & \varpi^2 - k_\perp^2 c_s^2 - k^2 v_A^2 & -k_\perp k_\parallel c_s^2 \\ 0 & -k_\perp k_\parallel c_s^2 & \varpi^2 - k_\parallel^2 c_s^2 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = 0$$

$v_A = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén velocity

$c_s = \sqrt{\gamma p_0/\rho_0}$ the sound speed

$k^2 = k_\perp^2 + k_\parallel^2$

giving dispersion relation: $(\varpi^2 - k_\parallel^2 v_A^2)[\varpi^4 - \varpi^2 k^2 (c_s^2 + v_A^2) + k^2 c_s^2 k_\parallel^2 v_A^2] = 0$

Wave Modes

- Alfven Wave: $\varpi^2 = \varpi_A^2 = k_\parallel^2 v_A^2$ incompressible, motion perpendicular to $\mathbf{B}_0$, magnetic perturbation is given by $\boldsymbol{b}_1 = \pm\sqrt{4\pi\rho_0}\,\boldsymbol{v}_1$ and phase speed given by the Alfven speed, V_A
- Fast Mode Wave: $\varpi^2 = \varpi_{\text{fast}}^2 = \frac{1}{2}k^2\left[v_A^2 + c_s^2 + \sqrt{(v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 k_\parallel^2/k^2}\right]$ also called the compressible Alfven Wave. The phase velocity is given by $v_A^2 + c_s^2 \geq (\varpi/k)^2 \geq v_A^2$ fastest for motion perpendicular to $\mathbf{B}_0$.
- For parallel propagation the wave is given by

$$\varpi^2 = \varpi_{\text{fast}}^2 = \frac{1}{2}k^2\left(v_A^2 + c_s^2 + |v_A^2 - c_s^2|\right)$$

In low-β plasma merges with the Alfven wave. For high-β plasma merges with the nonmagnetic sound wave

Wave Modes (Cont.)

- Slow Mode Wave: $\varpi^2 = \varpi_{\rm slow}^2 = \frac{1}{2}k^2\left[v_{\rm A}^2 + c_{\rm s}^2 - \sqrt{(v_{\rm A}^2 + c_{\rm s}^2)^2 - 4v_{\rm A}^2 c_{\rm s}^2 k_\parallel^2/k^2}\right]$
  with a phase speed of $0 \le (\varpi/k)^2 \le c_{\rm s}^2$ For perpendicular propagation, there is no restoring force, corresponding to a quasi-static equilibrium change
- For parallel propagation the phase velocity reaches its upper limit

$$\varpi_{\rm slow}^2 = \frac{1}{2}k^2\left(v_{\rm A}^2 + c_{\rm s}^2 - |v_{\rm A}^2 - c_{\rm s}^2|\right)$$

  If $V_A >\ C_S >$ the mode becomes the nonmagetic sound wave, otherwise the Alfven wave

- For all wave modes, this relation for phase velocity always holds:

$$v_{\rm fast} \ge v_{\rm A} \ge v_{\rm slow}$$

Waves in a Stratified System

- For perturbations in a stratified equilibrium $\rho_0(z)$ under the influence of gravity, magnetic fields and viscosity are neglected. Again doing a Fourier transfer gives

$$-i\varpi \boldsymbol{\omega}_1 = \frac{g}{\rho_0}\boldsymbol{e}_z \times i\boldsymbol{k}\rho_1 + 2i\boldsymbol{k}\cdot\boldsymbol{\Omega}\boldsymbol{v}_1$$

$$-i\varpi\rho_1 = -\rho_0' v_{1z},$$

- Using $\boldsymbol{g} = -g\boldsymbol{e}_z$ and curl with the relation $i\boldsymbol{k}\times\boldsymbol{\omega}_1 = k^2\boldsymbol{v}_1$

$$i\varpi k^2 \boldsymbol{v}_1 = -2i\boldsymbol{k}\cdot\boldsymbol{\Omega}\boldsymbol{\omega}_1 - \frac{1}{\rho_0}(gk_z\boldsymbol{k} - g\boldsymbol{e}_z k^2)\rho_1$$

which leads to dispersion relation

$$\varpi^2 k^2 + k_\perp^2 g\rho_0'/\rho_0 - 4(\boldsymbol{k}\cdot\boldsymbol{\Omega})^2 = 0.$$

Waves in a Stratified System (Cont.)

- Of the terms in that equation, the *Brunt-Vaisala* frequency, N is given by $N^2 = -g\rho_0'/\rho_0$
- The frequency of perturbations is given by

$$\varpi^2 = \frac{N^2 k_\perp^2 + 4(\boldsymbol{k} \cdot \boldsymbol{\Omega})^2}{k^2}$$

- In the case of a non-rotating plasma, all that is left is $\varpi = \pm N k_\perp / k, \quad \text{if } N^2 > 0, \text{ i.e., } g\rho_0' < 0$

  corresponding to a gravity wave with a frequency well under the sound speed, making the waves incompressible. This is called stable stratification because the lightest fluids are on top.

Rayleigh-Taylor Instability and Internal Waves

- In the case that $g\rho_0' > 0$ the heavier fluids are on top and the perturbation is not a propagating wave but instead grows exponentially and is the Rayleigh-Taylor instability
- Fast rotation will quickly stabilize the Rayleigh-Taylor instability
- In the case that N=0, the wave is the internal wave with frequency is given by

$$\varpi = \pm 2\boldsymbol{k} \cdot \boldsymbol{\Omega} / k.$$

Elsasser Fields and Alfven Time Normalization

- Since turbulence deals largely with incompressible plasma, the Alfven Mode is the most important linear MHD mode
- Writing the nonlinear MHD equations in terms of Elsasser Fields: $\boldsymbol{z}^{\pm} = \boldsymbol{v} \pm \frac{1}{\sqrt{4\pi\rho_0}}\boldsymbol{b}$
- The equation is simpler if normalized by the Alfven Time $\tau_{\mathrm{A}} = L/v_{\mathrm{A}}$

$$t/\tau_{\mathrm{A}} := t, \quad x/L := x, \quad b/B_0 := b, \quad p/(\rho_0 v_{\mathrm{A}}^2) := p,$$

where L is a convenient scale length, B_0 a typical magnetic field, and $v_{\mathrm{A}} = B_0/\sqrt{4\pi\rho_0}$ the corresponding Alfvén speed.

- The magnetic diffusivity is the inverse of the Lundquist number $S = v_{\mathrm{A}} L/\eta$
- This makes the Elsasser Field: $\boldsymbol{z}^{\pm} = \boldsymbol{v} \pm \boldsymbol{b}$

Elsasser Fields (Cont.)

- Combining the equation of motion and the magnetic induction equation while assuming incompressibility for the Elsasser field gives

$$\partial_t z^{\pm} + z^{\mp} \cdot \nabla z^{\pm} = -\nabla P + \tfrac{1}{2}(\nu + \eta)\nabla^2 z^{\pm} + \tfrac{1}{2}(\nu - \eta)\nabla^2 z^{\mp}$$

$$\nabla \cdot z^{\pm} = 0,$$

- Linearizing the field about a uniform magnetic field $\mathbf{B}_0$ and neglecting dissipation: $\partial_t z^{\pm} \mp \boldsymbol{B}_0 \cdot \nabla z^{\pm} = 0$

- z^- describes motion in the $\mathbf{B}_0$ direction and z^+ describes motion in the anti-$\mathbf{B}_0$ direction. There is only cross-coupling of z^- and z^+

Ideal Invariants in Terms of Elsasser Fields

- The invariants of Elsasser fields are

$$E = \tfrac{1}{4}\int dV\,\left[(z^+)^2 + (z^-)^2\right]$$

$$H^C = \tfrac{1}{4}\int dV\,\left[(z^+)^2 - (z^-)^2\right]$$

- Another important quantity is the difference between the kinetic and magnetic energies, the residual energy

$$E^R = \tfrac{1}{2}\int dV\,(\boldsymbol{v}^2 - \boldsymbol{b}^2) = \tfrac{1}{2}\int dV\,\boldsymbol{z}^+ \cdot \boldsymbol{z}^-$$