# Chapter 2: Magnetohydrodynamics
# Magnetohydrodynamic Turbulence
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# Conservation Laws-Fluid Invariants

- The momentum equation in conservation form

$$\partial_t \rho \boldsymbol{v} = \nabla \cdot \mathcal{T} + \rho \boldsymbol{g},$$

where $\mathcal{T} = \{T_{ij}\}$ is the total stress tensor, which can be written in the form

$$T_{ij} = -\left(p + \frac{B^2}{8\pi}\right)\delta_{ij} - \left(\rho v_i v_j - \frac{B_i B_j}{4\pi}\right) + \mu(\partial_i v_j + \partial_j v_i - \tfrac{2}{3}\delta_{ij}\,\nabla \cdot \boldsymbol{v})$$
$$= -P\delta_{ij} + R_{ij} + \sigma_{ij}^{(\mu)}.$$

$$\frac{d}{dt}\int_V \rho \boldsymbol{v}\, dV = \oint_S \mathcal{T} \cdot d\boldsymbol{S} + \int_V \rho \boldsymbol{g}\, dV$$

- Total Energy Density

$$\partial_t\left(\rho(\tfrac{1}{2}v^2 + u + \phi_g) + \frac{1}{8\pi}B^2\right) + \nabla \cdot \boldsymbol{F}^E = 0,$$

with the energy flux

$$\boldsymbol{F}^E = (\tfrac{1}{2}v^2 + h + \phi_g)\rho\boldsymbol{v} - \boldsymbol{\sigma}^{(\mu)} \cdot \boldsymbol{v} + \boldsymbol{q} + \frac{c}{4\pi}\boldsymbol{E} \times \boldsymbol{B}$$

where $h$ is the enthalpy,

$$h = u + \frac{p}{\rho} = \frac{\gamma}{\gamma - 1}\frac{p}{\rho}$$

# Total Energy Density

Multiply by ρv $$\rho \frac{d\boldsymbol{v}}{dt} \equiv \rho(\partial_t + \boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B} + \rho\boldsymbol{g} + \mu(\nabla^2\boldsymbol{v} + \tfrac{1}{3}\nabla\nabla\cdot\boldsymbol{v})$$ g=∇ Φ

Multiply by B $$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \eta\nabla^2\boldsymbol{B}.$$

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g=∇ Φ

Multiply by B

$$\partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = \eta\nabla^2\boldsymbol{B},$$

Add both to Energy Equation

$$\rho(\partial_t + \boldsymbol{v}\cdot\nabla)u + p\,\nabla\cdot\boldsymbol{v} = -\nabla\cdot\boldsymbol{q} + \boldsymbol{\sigma}^{(\mu)} : \nabla\boldsymbol{v} - \frac{1}{\sigma}\boldsymbol{j}^2$$

with the heat flux $\boldsymbol{q} = -\kappa\rho\nabla T$ (the colon denotes the dyadic product).

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with the heat flux $\boldsymbol{q} = -\kappa\rho\nabla T$ (the colon denotes the dyadic product).

You can now separate the energy equation into temporal and spatial derivatives, the left hand term represents the total Energy in the time derivative, changed by the energy flux moving in and out of the surface element

$$\partial_t\left(\rho(\tfrac{1}{2}v^2 + u + \phi_g) + \frac{1}{8\pi}B^2\right) + \nabla\cdot\boldsymbol{F}^E = 0,$$

where $h$ is the enthalpy,

$$h = u + \frac{p}{\rho} = \frac{\gamma}{\gamma-1}\frac{p}{\rho}$$

with the energy flux

$$\boldsymbol{F}^E = (\tfrac{1}{2}v^2 + h + \phi_g)\rho\boldsymbol{v} - \boldsymbol{\sigma}^{(\mu)}\cdot\boldsymbol{v} + \boldsymbol{q} + \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{B}$$

$$\frac{dE}{dt} = -\oint_S d\boldsymbol{S}\cdot\boldsymbol{F}^E.$$

$$E = \int dV\left(\rho(\tfrac{1}{2}\rho v^2 + u + \phi_g) + \frac{1}{8\pi}B^2\right)$$

# The Energy Equation

- Including incompressibility and introducing the dissipative terms:

$$\frac{dE}{dt} = -\oint_S d\boldsymbol{S} \cdot \boldsymbol{F}^E - D^E$$

$$E = \int dV \left( \frac{1}{2}\rho v^2 + \rho\phi_g + \frac{1}{8\pi} B^2 \right)$$

$\boldsymbol{F}^E$ contains no dissipative terms,

$$\boldsymbol{F}^E = (\tfrac{1}{2}\rho v^2 + p + \rho\phi_g)\boldsymbol{v} + \frac{1}{4\pi}\boldsymbol{B} \times (\boldsymbol{v} \times \boldsymbol{B})$$

and $D^E$ comprises the energy dissipation,

$$D^E = \int dV \left( \frac{1}{\sigma}\boldsymbol{j}^2 + \mu\boldsymbol{\omega}^2 \right).$$

# Cross-Helicity

$$H^C = \int \boldsymbol{v} \cdot \boldsymbol{B} \, dV$$

Multiply by B/ρ $$\rho \frac{d\boldsymbol{v}}{dt} \equiv \rho(\partial_t + \boldsymbol{v} \cdot \nabla)\boldsymbol{v} = -\nabla p + \frac{1}{c}\boldsymbol{j} \times \boldsymbol{B} + \rho \boldsymbol{g} + \mu(\nabla^2 \boldsymbol{v} + \tfrac{1}{3}\nabla\nabla \cdot \boldsymbol{v})$$ g=∇ Φ

Multiply by v $$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \eta \nabla^2 \boldsymbol{B}.$$

$$\boldsymbol{F}^C = \boldsymbol{v} \times (\boldsymbol{v} \times \boldsymbol{B}) + \left(\phi_g + \frac{\gamma}{\gamma - 1}\frac{p}{\rho}\right)\boldsymbol{B}$$

$$D^C = (\nu + \eta) \int_V dV \sum_{i,j} \partial_i B_j \, \partial_i v_j$$

$$\frac{dH^C}{dt} = -\oint_S \boldsymbol{F}^C \cdot d\boldsymbol{S} - D^C$$

# Magnetic Invariants-Flux

- Magnetic Flux defined by $\Phi = \int_S \boldsymbol{B} \cdot d\boldsymbol{S}$
- Applying Stokes' Theorem to the induction eq.:

$$\partial_t \boldsymbol{B} - \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) = \eta \nabla^2 \boldsymbol{B}, \qquad c(\nabla \times \boldsymbol{j}) = \nabla^2 \boldsymbol{B} + \nabla(\nabla \cdot \boldsymbol{B})$$

$$\int_S \partial_t \boldsymbol{B} \cdot d\boldsymbol{S} = \oint_l (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l} - \frac{c}{\sigma} \oint_l \boldsymbol{j} \cdot d\boldsymbol{l}.$$

- The v x B term is the flux through the surface

$$\oint (\boldsymbol{v} \times \boldsymbol{B}) \cdot d\boldsymbol{l}\, dt = -\oint \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{l})\, dt = \int_{dS} \boldsymbol{B} \cdot d\boldsymbol{S}$$

$$\frac{d\Phi}{dt} = \int_S \partial_t \boldsymbol{B} \cdot d\boldsymbol{S} + \oint_l \boldsymbol{B} \cdot (\boldsymbol{v} \times d\boldsymbol{l}) = -\frac{c}{\sigma} \oint_l \boldsymbol{j} \cdot d\boldsymbol{l}.$$

# Helicity

- Inserting Farday's Law into the helicity equation and using gauge $E = -\partial_t A/c$:

$$\partial_t \boldsymbol{B} = -c\,\nabla \times \boldsymbol{E} \qquad H^M = \int_V \boldsymbol{A}\cdot\boldsymbol{B}\,dV$$

$$\int \partial_t(\boldsymbol{A}\cdot\boldsymbol{B})\,dV = \int (\boldsymbol{B}\cdot\partial_t\boldsymbol{A} + \boldsymbol{A}\cdot\partial_t\boldsymbol{B})\,dV$$

$$= -2c\int \boldsymbol{E}\cdot\boldsymbol{B}\,dV + c\oint(\boldsymbol{A}\times\boldsymbol{E})\cdot d\boldsymbol{S}$$

- and then and using the boundary condition $B_n$=0 and Ohm's Law: $\boldsymbol{E} + \frac{1}{c}\boldsymbol{v}\times\boldsymbol{B} = \frac{1}{\sigma}\boldsymbol{j}$

$$c\oint(\boldsymbol{A}\times\boldsymbol{E})\cdot d\boldsymbol{S} \longrightarrow -\oint(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{S}$$

$$\oint(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{S}\,dt = \int_{dV}\boldsymbol{A}\cdot\boldsymbol{B}\,dV$$

- which again is 0 for infinite conductivity

$$\frac{dH^M}{dt} = \int \partial_t(\boldsymbol{A}\cdot\boldsymbol{B})\,dV + \oint(\boldsymbol{A}\cdot\boldsymbol{B})\boldsymbol{v}\cdot d\boldsymbol{S} = -\frac{2c}{\sigma}\int dV\,\boldsymbol{j}\cdot\boldsymbol{B}.$$

# Linear Waves in a Homogeneous Magnetized System

- The basic elements of turbulence are formed by waves in the plasma, oscillating about an average state
- In a homogeneous system defined by $p_0$, $\rho_0$, embedded in magnetic field $\mathbf{B}_0$, for small changes $\tilde{p} \ll p_0, \tilde{\boldsymbol{b}} \ll \boldsymbol{B}_0$ the MHD equations are linear
- Performing a Fourier transform $\tilde{\boldsymbol{v}}(\boldsymbol{x}, t) = \boldsymbol{v}_1 \exp(i\boldsymbol{k}\cdot\boldsymbol{x} - i\varpi t)$ in space and time yields

$$\rho \frac{d\boldsymbol{v}}{dt} \equiv \rho(\partial_t + \boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \frac{1}{c}\boldsymbol{j}\times\boldsymbol{B} + \rho\boldsymbol{g} + \mu(\nabla^2\boldsymbol{v} + \tfrac{1}{3}\nabla\nabla\cdot\boldsymbol{v}) \longrightarrow -i\varpi\rho_0\boldsymbol{v}_1 = -i\boldsymbol{k}p_1 + \frac{1}{4\pi}(i\boldsymbol{k}\times\boldsymbol{b}_1)\times\boldsymbol{B}_0 - \mu k^2\boldsymbol{v}_1$$

$$\partial_t\boldsymbol{B} - \nabla\times(\boldsymbol{v}\times\boldsymbol{B}) = \eta\nabla^2\boldsymbol{B} \longrightarrow -i\varpi\boldsymbol{B}_1 = i\boldsymbol{k}\times(\boldsymbol{v}_1\times\boldsymbol{B}_0) - \eta k^2\boldsymbol{B}_1,$$

$$\partial_t p + \boldsymbol{v}\cdot\nabla p + \gamma p\nabla\cdot\boldsymbol{v} = 0 \longrightarrow -i\varpi p_1 = -i\gamma p_0\boldsymbol{k}\cdot\boldsymbol{v}_1.$$

# Waves in a Homogeneous Mangetized System (Cont.)

- Performing substitutions on $v_1$, $B_1$, and $p_1$, these three equations become one equation

$$\varpi^2 \rho_0 \boldsymbol{v}_1 = \left( \frac{\boldsymbol{B}_0 \times (\boldsymbol{k} \times \boldsymbol{B}_0)}{4\pi} + \gamma p_0 \boldsymbol{k} \right) \boldsymbol{k} \cdot \boldsymbol{v}_1 - \frac{1}{4\pi} \boldsymbol{k} \cdot \boldsymbol{B}_0 (\boldsymbol{k} \times \boldsymbol{v}_1) \times \boldsymbol{B}_0$$

- The **k·v** term represents longitudinal, compressible waves and the **k** x **v** term represents transverse waves
- By choosing the coordinate system $\boldsymbol{B}_0 = B_0 \boldsymbol{e}_z, \boldsymbol{k} = k_\perp \boldsymbol{e}_y + k_\parallel \boldsymbol{e}_z$ and writing the equation in matrix form

$$\begin{pmatrix} \varpi^2 - k_\parallel^2 v_A^2 & 0 & 0 \\ 0 & \varpi^2 - k_\perp^2 c_s^2 - k^2 v_A^2 & -k_\perp k_\parallel c_s^2 \\ 0 & -k_\perp k_\parallel c_s^2 & \varpi^2 - k_\parallel^2 c_s^2 \end{pmatrix} \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix} = 0$$

$v_A = B_0/\sqrt{4\pi\rho_0}$ is the Alfvén velocity

$c_s = \sqrt{\gamma p_0/\rho_0}$ the sound speed

$k^2 = k_\perp^2 + k_\parallel^2$

giving dispersion relation: $(\varpi^2 - k_\parallel^2 v_A^2)[\varpi^4 - \varpi^2 k^2 (c_s^2 + v_A^2) + k^2 c_s^2 k_\parallel^2 v_A^2] = 0$

# Wave Modes

- Alfven Wave: $\varpi^2 = \varpi_A^2 = k_\parallel^2 v_A^2$ incompressible, motion perpendicular to $\mathbf{B}_0$, magnetic perturbation is given by $\boldsymbol{b}_1 = \pm\sqrt{4\pi\rho_0}\,\boldsymbol{v}_1$ and phase speed given by the Alfven speed, $V_A$
- Fast Mode Wave: $\varpi^2 = \varpi_{\text{fast}}^2 = \frac{1}{2}k^2\left[v_A^2 + c_s^2 + \sqrt{(v_A^2 + c_s^2)^2 - 4v_A^2 c_s^2 k_\parallel^2/k^2}\right]$ also called the compressible Alfven Wave. The phase velocity is given by $v_A^2 + c_s^2 \geq (\varpi/k)^2 \geq v_A^2$ fastest for motion perpendicular to $\mathbf{B}_0$.
- For parallel propagation the wave is given by

$$\varpi^2 = \varpi_{\text{fast}}^2 = \frac{1}{2}k^2\left(v_A^2 + c_s^2 + |v_A^2 - c_s^2|\right)$$

In low-β plasma merges with the Alfven wave. For high-β plasma merges with the nonmagnetic sound wave

# Wave Modes (Cont.)

- Slow Mode Wave: $\varpi^2 = \varpi_{\rm slow}^2 = \frac{1}{2}k^2\left[v_{\rm A}^2 + c_{\rm s}^2 - \sqrt{(v_{\rm A}^2 + c_{\rm s}^2)^2 - 4v_{\rm A}^2 c_{\rm s}^2 k_\parallel^2/k^2}\right]$
  with a phase speed of $0 \le (\varpi/k)^2 \le c_{\rm s}^2$ For perpendicular propagation, there is no restoring force, corresponding to a quasi-static equilibrium change
- For parallel propagation the phase velocity reaches its upper limit

  $$\varpi_{\rm slow}^2 = \frac{1}{2}k^2(v_{\rm A}^2 + c_{\rm s}^2 - |v_{\rm A}^2 - c_{\rm s}^2|)$$

  If $V_A > C_S$ the mode becomes the nonmagetic sound wave, otherwise the Alfven wave
- For all wave modes, this relation for phase velocity always holds: $v_{\rm fast} \ge v_{\rm A} \ge v_{\rm slow}$

# Waves in a Stratified System

- For perturbations in a stratified equilibrium $\rho_0(z)$ under the influence of gravity, magnetic fields and viscosity are neglected. Using hydrostatic equilibrium $\nabla p_0 = \boldsymbol{g}\rho_0$ Again doing a Fourier transfer gives

$$\partial_t \boldsymbol{\omega} + \boldsymbol{v}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\boldsymbol{v} = \frac{1}{c\rho_0}[\boldsymbol{B}_0\cdot\nabla\tilde{\boldsymbol{j}} - \tilde{\boldsymbol{j}}\cdot\nabla(\boldsymbol{B}_0 + \tilde{\boldsymbol{B}})] + \frac{1}{\rho_0}\nabla\tilde{\rho}\times\boldsymbol{g} + 2\boldsymbol{\Omega}\cdot\nabla\boldsymbol{v} + \nu\nabla^2\boldsymbol{\omega} \longrightarrow -i\varpi\boldsymbol{\omega}_1 = \frac{g}{\rho_0}\boldsymbol{e}_z\times i\boldsymbol{k}\rho_1 + 2i\boldsymbol{k}\cdot\boldsymbol{\Omega}\boldsymbol{v}_1$$

$$\boldsymbol{g} = -g\boldsymbol{e}_z$$

$$\tilde{\rho}/\rho_0 \simeq -\tilde{T}/T_0 \qquad \partial_t\tilde{T} + \boldsymbol{v}\cdot\nabla(\tilde{T} + T_0) = \kappa\nabla^2\tilde{T} \longrightarrow -i\varpi\rho_1 = -\rho_0' v_{1z},$$

- Using the curl with the relation $i\boldsymbol{k}\times\boldsymbol{\omega}_1 = k^2\boldsymbol{v}_1$

$$-i\varpi\boldsymbol{\omega}_1 = \frac{g}{\rho_0}\boldsymbol{e}_z\times i\boldsymbol{k}\rho_1 + 2i\boldsymbol{k}\cdot\boldsymbol{\Omega}\boldsymbol{v}_1 \longrightarrow i\varpi k^2\boldsymbol{v}_1 = -2i\boldsymbol{k}\cdot\boldsymbol{\Omega}\boldsymbol{\omega}_1 - \frac{1}{\rho_0}(gk_z\boldsymbol{k} - g\boldsymbol{e}_z k^2)\rho_1$$

- which leads to dispersion relation $\varpi^2 k^2 + k_\perp^2 g\rho_0'/\rho_0 - 4(\boldsymbol{k}\cdot\boldsymbol{\Omega})^2 = 0.$

# Waves in a Stratified System (Cont.)

- Of the terms in that equation, the *Brunt-Vaisala* frequency, N is given by $N^2 = -g\rho_0'/\rho_0$
- The frequency of perturbations is given by

$$\varpi^2 = \frac{N^2 k_\perp^2 + 4(\boldsymbol{k}\cdot\boldsymbol{\Omega})^2}{k^2}$$

- In the case of a non-rotating plasma, all that is left is $\varpi = \pm N k_\perp / k, \quad \text{if } N^2 > 0, \text{ i.e., } g\rho_0' < 0$

  corresponding to a gravity wave with a frequency well under the sound speed, making the waves incompressible. This is called stable stratification because the lightest fluids are on top.

# Rayleigh-Taylor Instability and Internal Waves

- In the case that $g\rho_0' > 0$ the heavier fluids are on top and the perturbation is not a propagating wave but instead grows exponentially and is the Rayleigh-Taylor instability

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- Fast rotation will quickly stabilize the Rayleigh-Taylor instability
- In the case that N=0, the wave is the internal wave with frequency is given by

$$\varpi = \pm 2\boldsymbol{k} \cdot \boldsymbol{\Omega} / k.$$

# Elsasser Fields and Alfven Time Normalization

- Since turbulence deals largely with incompressible plasma, the Alfven Mode is the most important linear MHD mode
- Writing the nonlinear MHD equations in terms of Elsasser Fields: $z^{\pm} = v \pm \frac{1}{\sqrt{4\pi\rho_0}} b$
- The equation is simpler if normalized by the Alfven Time $\tau_A = L/v_A$

$$t/\tau_A := t, \quad x/L := x, \quad b/B_0 := b, \quad p/(\rho_0 v_A^2) := p,$$

where $L$ is a convenient scale length, $B_0$ a typical magnetic field, and $v_A = B_0/\sqrt{4\pi\rho_0}$ the corresponding Alfvén speed.

- The magnetic diffusivity is the inverse of the Lundquist number $S = v_A L/\eta$
- This makes the Elsasser Field: $z^{\pm} = v \pm b$

# Elsasser Fields (Cont.)

- Combining the equation of motion and the magnetic induction equation while assuming incompressibility for the Elsasser field gives

$$\partial_t z^{\pm} + z^{\mp} \cdot \nabla z^{\pm} = -\nabla P + \tfrac{1}{2}(\nu + \eta)\nabla^2 z^{\pm} + \tfrac{1}{2}(\nu - \eta)\nabla^2 z^{\mp}$$

$$\nabla \cdot z^{\pm} = 0,$$

- Linearizing the field about a uniform magnetic field $\mathbf{B}_0$ and neglecting dissipation: $\partial_t z^{\pm} \mp \boldsymbol{B}_0 \cdot \nabla z^{\pm} = 0$

- $z^-$ describes motion in the $\mathbf{B}_0$ direction and $z^+$ describes motion in the anti-$\mathbf{B}_0$ direction. There is only cross-coupling of $z^-$ and $z^+$

# Ideal Invariants in Terms of Elsasser Fields

- The invariants of Elsasser fields are

$$E = \frac{1}{4} \int dV \left[ (z^+)^2 + (z^-)^2 \right]$$

$$H^C = \frac{1}{4} \int dV \left[ (z^+)^2 - (z^-)^2 \right]$$

- Another important quantity is the difference between the kinetic and magnetic energies, the residual energy

$$E^R = \frac{1}{2} \int dV \, (\boldsymbol{v}^2 - \boldsymbol{b}^2) = \frac{1}{2} \int dV \, \boldsymbol{z}^+ \cdot \boldsymbol{z}^-$$

# What I've Learned from this class

- The MHD equations are extremely nuanced, and the different assumptions and simplifications needed to handle them must be very carefully considered
- When doing modeling, it is important to fully understand the problem that is being solved, so that the right boundary conditions and assumptions can be used
- While it is necessary to be very careful with the math, it is also important to not get so caught up in the math that the physical meaning of the terms is lost. The math and the physics are almost impossible to understand without one another